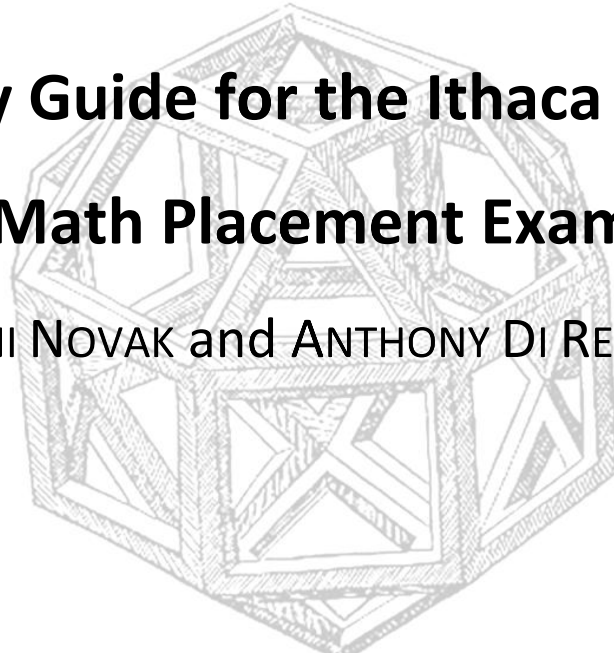
TWELVE MATHEMATICAL CONCEPTS:

A Study Guide for the Ithaca College

Math Placement Exam

DANI NOVAK and ANTHONY DI RENZO



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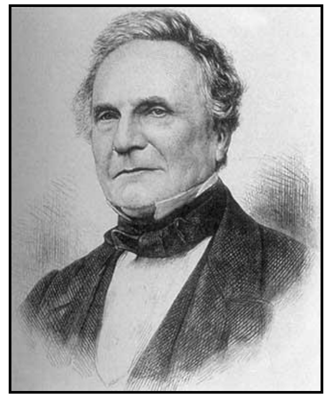


In a letter to Alfred Lord Tennyson, Charles Babbage criticized a line in the poem "The Vision of Sin": Tennyson was incorrect to say, "*Every minute a man dies, Every minute a man is born.*" Statistically, the world's population was not in a state of equilibrium but constantly increasing. Tennyson should have written: "*Every minute dies a man, And one and a sixteenth is born.* Strictly speaking, Babbage added, the exact figure was 1.067, "but something must, of course, be conceded to the laws of metre."

This story supports a common prejudice. Supposedly, writers can't count, and mathematicians can't write. But like most prejudices, this one is false. Words and numbers complement each other. Both symbol systems encode and decode our world. That is why the Department of Math and the Department of Writing at Ithaca College have created this study guide *Twelve Mathematical Concepts.*

By defining and illustrating basic mathematical concepts, this guide will help incoming students prepare for the college's Math Placement Exam. For this reason, it contains many practical exercises. But this guide also hopes to stimulate an interest in math, to explore its beauty, and to demonstrate its relationship to the arts and humanities. Accordingly, this guide also includes historical anecdotes, conceptual illustrations, and philosophical meditations. We believe this material makes math instruction less abstract and more holistic.

"Mathematics," Albert Einstein observed, "is the poetry of logical ideas, just as poetry can be as elegant as mathematical proofs." Putting our words and numbers at your service, we hope you will find this guide useful and engaging.

DANI NOVAK | ANTHONY DI RENZO

Concept: Number Representations

Numbers are fundamental to math, of course, but there are many ways to write and represent them.

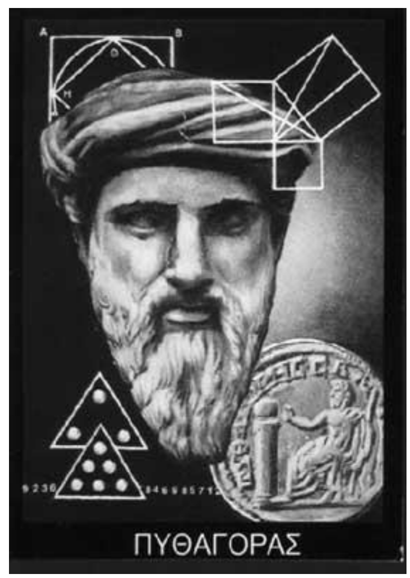
Symbols: Western math uses ten distinct symbols to represent all numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Fraction: A *quotient* or partial representation of number (e.g. "¾").

Decimal: Notation built on the *base-ten* numeral system. For example 512 translates into 5*10*10 + 1*10 + 2. 632.53 = 6*10*10 + 3*10 + 2 + 5/10 + 3/(10*10)

Binary: Notation built on *base-two* numeral system, specifically 0 and 1. To illustrate, 110.1 in binary is 2*2+2+0+1/2 = 6.5 in decimal

Percent: Same as decimal except we add two 0's and the %. ¾ in fraction is also 75/100, which equals 0.75 in decimal (or **75%**).



Background: Elam, an ancient civilization located in what is now southwest Iran, possibly originated the decimal system, but the earliest recorded use of decimal fractions (circa 2800 BC) occurred in the Indus Valley. Although the Egyptians invented a forerunner of fractions as early as 1000 BC, the Greeks perfected and regularly used them around 530 BC, thanks to the Pythagoreans. Three centuries later, Jain mathematicians in India wrote the *Sthananga Sutra*, which contains work on the theory of numbers, arithmetical operations, and operations with fractions.

Philosophy: Numbers are fundamental to human consciousness. Clinical research proves the brain is actually hard wired for math. For example, as mathematician John Allen Paulos suggests, because we perceive *parts* and *wholes* in nature we understand fractions.

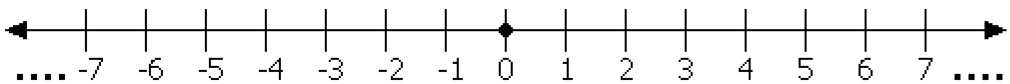
Application: If you buy a $15 book and pay an additional sales tax of 8%, how much change will you get from $20 bill?

Answer: The total is $15*1.08 = $16.20, so the change is $20.00 – $16.20 = $3.80.

Concept: Dimension and Coordinates

Definition: A space's **dimension** consists of the minimum amount of numbers (called *coordinates*) needed to specify every point within it.

Definition: A **number line,** a *one-dimensional* picture of a line, shows integers as specially marked, evenly spaced points on a continuum. This line includes all real numbers, continuing forever in each direction from zero. John Wallis, the Puritan mathematician and cryptographer, created this graph in the 17th century.



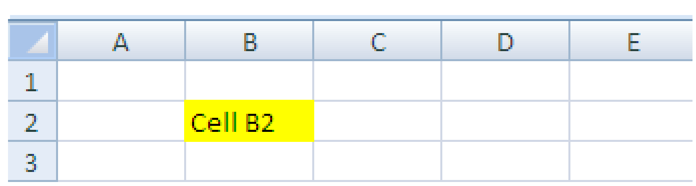
Definition: The **Cartesian coordinate system,** named after René Descarte, is a *two-dimensional* picture of **plane**, consisting of two intersecting number lines. Forming a cross, this rectangular graph can determine any point in a plane by using two numbers: the *x-coordinate* and the *y-coordinate*.

Remarks: Examples of the *n-dimensional system*, the number line is limited to 1 dimension, the Cartesian coordinate system to 2 or 3. General relativity, however, uses a 4-dimensional system, and current theories in physics talk about a 10-dimensional universe.

Background: Developed independently in 1637 by Descartes and Pierre de Fermat (although Fermat never published his discovery), the coordinate system combines two complementary aspects of human reasoning: *logic* and *intuition*. This useful breakthrough influenced the development of analytic geometry, calculus, and cartography.

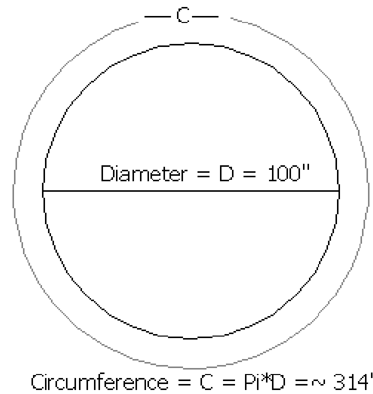
Philosophy: Descartes mathematical insights resulted from vivid dreams. When facing problems, therefore, use both sides of your brain to solve them. Gut feeling gives subjective meaning and direction while critical analysis provides objective method and perspective. The best thinking balances both.

Application: Spreadsheet design uses a coordinate system. Every cell has two coordinates. The vertical is a number, the horizontal a letter.



Concept: Pi

<u>Definition:</u> **Pi** or **π**, a mathematical constant, represents the ratio of any circle's *circumference* to its *diameter*. This is the same as the ratio of a circle's *area* to the square of its *radius*, approximately **3.14159**.



<u>Remarks</u>: 1. The diameter (d=*2r) of a circle is twice its radius (r. 2). The area of a circle is Pi*r*r or Pi*r^2.

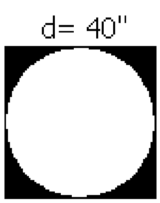
<u>Background</u>: As a rough concept, Pi was familiar to ancient Babylonian, Egyptian, Indian, and Greek philosophers, but the Sicilian Archimedes was first mathematician to calculate Pi rigorously. Perhaps the most beautiful reference to Pi occurs in the Bible, where a Hebrew chronicler describes the round basin located in front of the Temple of Jerusalem.

"And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass It about." (I Kings 7, 23)

Here the measure of Pi equals 3.

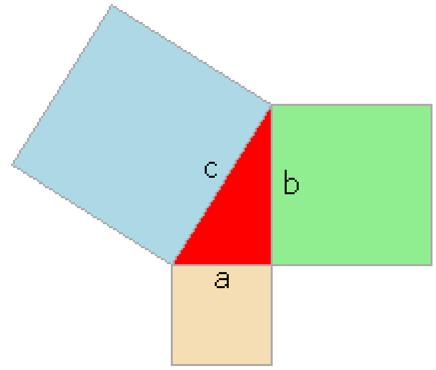
<u>Philosophy</u>: Pi is a *transcendental* number. No equation can ever define it. For the ancients, Pi represented the wholeness of the cosmos because a circle has no beginning and no end. Even so, every circle, no matter how infinite, has a single center. The same insight applies to the mind. We think we start our journey to knowledge in school, but actually, it has no beginning and no end. We each have an inner center but it is impossible to define it with words. The scope of our mind is as infinite as Pi.

<u>Application</u>: If flowers are planted around a circular pool within a square of length 40'', what is the area of the garden? See Picture.



<u>Answer</u>: The area of the square is d*d (1600 in^2) and the area of the pool is $Pi*20^2$ (~1256 in^2).The area of the garden, therefore, is the difference, which is (d^2 – $Pi*r^2$) or 344 in^2

Concept: Pythagoras Theorem

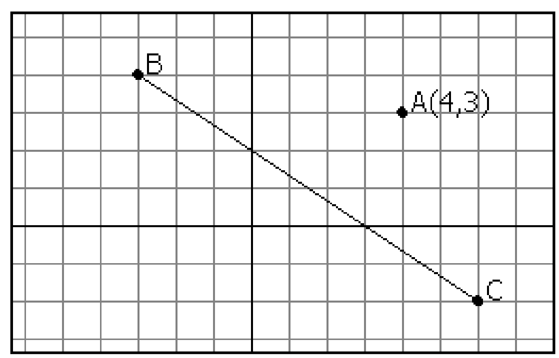


Definition: The square of the *hypotenuse* of a right triangle equals the sum of the squares on the other two sides. The pink and green squares have the same combined area as the blue square.

Remark: A **Pythagorean triple** consists of three positive integers *a*, *b*, and *c*, such that *a2* + *b2* = *c2*. Try this formula with the sequence 3, 4, 5.

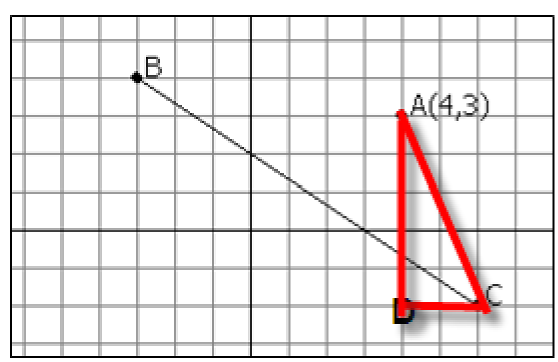
Background: Named after the Greek mathematician and philosopher Pythagoras, this may be the simplest, deepest, and most useful theorem ever. Five-thousand-year-old megalithic monuments in Egypt incorporate right triangles with integer sides. Bartel Leendert van der Waerden thought the ancients discovered these triples algebraically.

Philosophy: This theorem holds true for *all* right triangle regardless of the lengths of their sides. Some laws in the universe are eternally true, Pythagoras believed, and when we follow these laws, we are content. "No man is free," he declared, "unless he can command himself." According to legend, Pythagoras was so happy when he discovered this theorem that he sacrificed 100 oxen to the Muses; but numbers, he preached, rule Platonic forms, create gods and demons, and determine the music of the spheres.



Application: Two villages, located at points B and C, both use a well at point A. Assuming the grid units are miles, calculate the distance between A and C.

Answer: By imagining the red right triangle, we conclude he square of the distance between the point A and C is 2^2 + 5^2 = 29, so the distance between A and C is 29^0.5 or about 5.4 miles



Concept: Algebra

Definition: **Algebra** studies *structure*, *relation*, and *quantity*. By substituting concrete numbers with symbols, it generalizes arithmetic.

Background: Although the ancient Babylonians experimented with a form of algebra some 3,000 years ago, the Arabs perfected this specialized branch of mathematics when the Persian Muhammad ibn Mūsā al-khwārizmī published his great treatise *Al-Jabr* (820 AD). *Al-jabr* means "reunion." Because it balances both sides of an equation, algebra, the mathematician Viète claimed, "leaves no problem unsolved."

Philosophy: Bridging the concrete and abstract, algebra created *analytical thinking.* Fascinated by the whole, classical Greek math was *contemplative*. In contrast, Arab math was concerned with parts; it was *critical* and *empirical*. Algebra breaks down a complex phenomenon into its subcomponents and reconstructs it. Adopting the same technique in his *Meditations* (1641), René Descartes laid the foundation for both modern philosophy and the scientific method.

Applications:

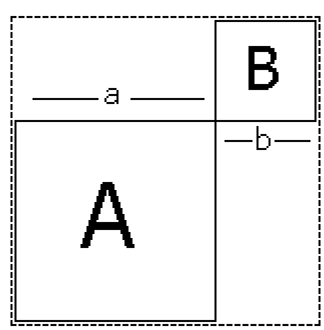
1. If you buy a book for x dollars (where x is no more than 10) and pay an additional sales tax of 8%, how much change will you get from $20 bill?

Answer: The book's price after tax is 1.08*x; hence the change is 20 – 1.08*x. Since x is a *variable*, the answer will depends on that variable, as in the above table.

2. Two square boxes A and B, with sides lengths a and b, are packed in a bigger box (as shown in the picture). What is the area of the big box?

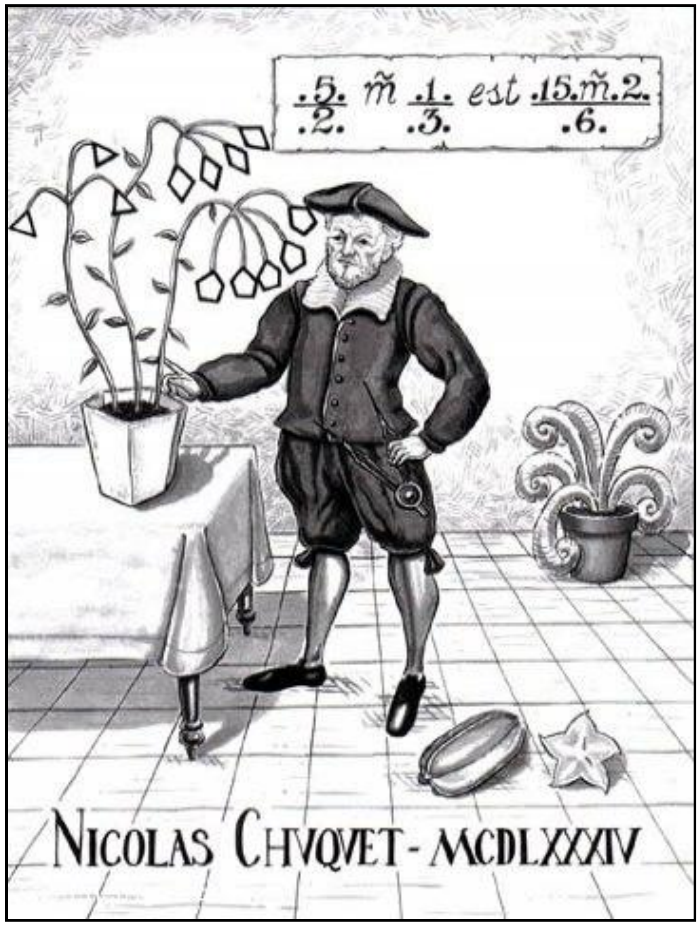
x	1	2	3	4	5	6	7	8	9	10
y	18.92	17.84	16.76	15.68	14.6	13.52	12.44	11.36	10.28	9.2

Answer: (a+b)^2 = (a+b)*(a+b) = a*a+a*b+b*a+b*b=a^2+2*a*b+b^2



Concept: Exponents

Definition The **exponent operation** originally meant *repeated multiplication*, just as multiplication means repeated addition. For any natural (positive and whole) number n, therefore, a^n = a*a*a… n times for any number a.



Remark: An exponent's definition is not limited to natural numbers. For example $a^{-n}=1/a^{n}$.   Also $a^{m/n}$ = the nth root of a^m.

Background: French mathema-tician Nicolas Chuquet (1445-48) used an early form of exponential notation, later adapted and improved by Henricus Gram-mateus and Michael Stifel. This groundbreaking symbol system generated numerous *exponential rules* that can be proven logically. (For example, a^{m+n} = a^m*a^n.) Schools around the world now teach these rules to students.

Philosophy: Exponents may be "just" notations but they allow us to communicate complex ideas in a way that previously would have been impossible. The introduction of notation spurred scientists and mathematicians to develop exponential rules. Nevertheless, human beings did not invent these rules but discovered them. Their validity lies in pure logic, but they remain true even beyond the Milky Way. Similarly, once consciousness develops, we create social rules to validate our connection to members of our community, even the stranger within our midst. Like words, numbers can be metaphors, but their poetry is literally true. According to Einstein, mathematical proofs suggest that the mind mirrors the cosmos.

Application: Show the logic behind the rule: $a^m * a^n = a^{m+n}$

Answer: Using the notation, $a^m * a^n$ = a*a*a…m times *a*a*a… n times so it must be a*a*a…(m+n) times, or a^{m+n}.

Concept: Logarithm

Definition: A **logarithm** states the *power* by which a *base* (usually 10) must be raised to produce a given number:

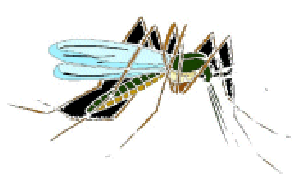
$$\text{if } b^y = x, \quad \text{then} \quad \log_b(x) = y\,.$$

Logarithms express the *ratio* of related numbers. By converting arithmetical progression into geometric progression, they make multiplication and division as simple as addition and subtraction. Very useful, indeed!

Example: The logarithm of 1000 to the base 10 is 3 because 10 x 10 x 10 = 1000. Likewise, the base 2 logarithm of 32 is 5, since 2 to the 5^{th} power is 32.

Background: Lord John Napier of Scotland, scientist and magician, first propounded logarithms in his landmark book *Mirifici Logarithmorum Canonis Descriptio* (1614). During the Enlightenment, they contributed to the advance of science, especially of astronomy, by streamlining difficult calculations. Before the advent of calculators and computers, logarithms were essential to surveying, navigation, and other branches of practical mathematics. Today, geologists measure earthquakes on the logarithmic Richter scale.

Philosophy: Logarithms are the *inverse* or opposite of exponentials, just as subtraction is the opposite of addition and division is the opposite of multiplication. Logs "undo" exponentials. By applying the log function to its inverse, one arrives at an exponent called the *identity function*. Intellectual debate works the same way. “The opposite of a trivial truth is false,” said Niels Bohr, “but the opposite of a great truth is another great truth.”

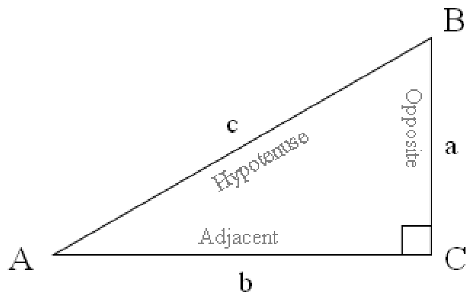


Application: A mosquito's buzz generates a *decibel rating* of 40 dB. Normal conversation rates 60 dB. How many times more intense is normal conversation than a mosquito's *buzz*?

Answer: 60 – 40 = 20 dB or two Bel. Since 10^2=100 (Or Log_{10} 100 = 2), normal conversation is about 100 times louder than a mosquito's buzz.

Concept: Trigonometry and Proportion

Definition: **Trigonometry** (from Greek *Τριγονομετρία* "*tri* = three" + "*gon* = angle" + *metr[y] = to measure")* deals with *ratios* of the *sides* of *right triangles.*



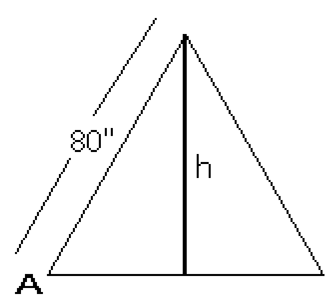
SOH-CAH-TOA.

Sine = **O**pposite ÷ **H**ypotenuse
Cosine = **A**djacent ÷ **H**ypotenuse
Tangent = **O**pposite ÷ **A**djacent

Background: First developed as a navigation method, trigonometry emerged over 4,000 years ago in ancient Egypt, Mesopotamia and the Indus Valley. But the Greek Hipparchus (circa 150 BC) compiled the first trigonometric table using the sine for solving triangles. With this table, Hipparchus became the greatest astronomer of antiquity, mapping the stars and predicting solar eclipses.

Philosophy: Many students associate trigonometry with memorizing meaningless formulas. Actually, it provides a firm framework for the art of proportion and applies to many practical everyday activities, from carpentry to space travel. Trigonometry blaze many trails—provided we remain patient and persevere. As W.S. Anglin observes: "Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness."

Application: A carpenter installs a stabilizing metal rod under a table top shaped like an equilateral triangle, with side lengths of 80 inches. How long is the rod?



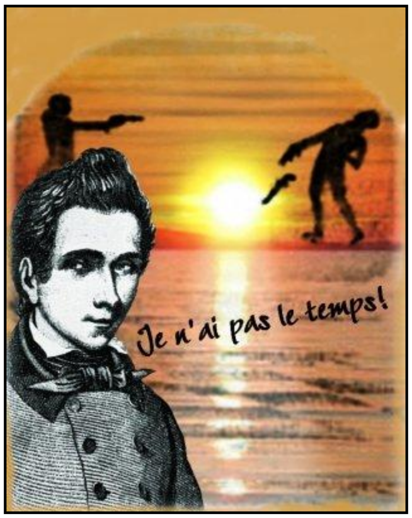
Answer: Since the sum of the angles of every triangle is 180 degrees, and all the angle of the equilateral triangle are equal, it follows that A=60 degrees and hence h/80 = Sine(60)~0.87 so h=80''*Sin(60)~80*0.87=69.6''

Concept: Quadratic Equations

Definition: The **quadratic** is a special case of a *polynomial equation* of the nth degree: ax+b=0 (linear, 1st degree), ax2+bx+c=0 (quadratic, 2nd degrees), ax3+bx2+cx+d=0 (cubic, 3rd degree), etc. Here's the quadratic solution . . .

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic equations derive their name from *quadratus* (Latin for "square") because the variable in the leading term is always squared.

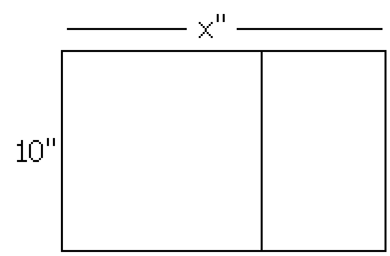


Background: The Persian poet-mathematician Omar Khayyám (1048–1131) first theorized about cubic equations. Five centuries later, Scipione del Ferro (1465-1526) and others blazed the way to a general formula. When Lodovico Ferrari, solved the quartic (4th degree equation) in 1540, his break-through inspired a quest for the Holy Grail of higher degree polynomials; but in 1824, Niels Henrik Abel (1802-1829) proved this was an illusion. Évariste Galois (1811-1832) concluded the same in a 30-page manuscript written the night before his fatal duel.

Philosophy: Sometimes we must prove something is impossible to achieve. Knowing our limitations is the beginning of wisdom and the foundation of science. Mathematicians should remember Reinhold Niebuhr's famous prayer: "God grant me the serenity, to accept the things I cannot change, the courage to change the things I can, and the wisdom to know the difference."

Application: During a renovation, a master carpenter divided a formal parlor into two smaller rooms: one shaped like a square, the other proportional to the original room. If the width of the square is 10 feet, what is the length (x) of the original room? (See picture.)

Solution: The sides of the smaller rectangle are 10'' and (x-10)''. To be proportional to the original square, the equation should be: x/10 = 10/(x – 10). This yields the quadratic equation of $x^2 - 10*x - 100 = 0$. Using the quadratic formula the positive solution of this equation is $x=10(1+5^{0.5})/2 \sim 16.2$'' (Incidentally, the proportion x/10 ~ 1.62 is called the *golden mean*.)



Concept: System of Equations

Definition: a **system of linear equations** (or **linear system**) is a collection of linear equations involving the same set of variables. For example, the system of equations in dimension 2:
$3x + 2y = 4$
$5x – 4y = 3$
has the unique solution x=1 and y=0.5

Background: Methods of solving linear equations belong to the field of Linear Algebra. The history of modern linear algebra dates back to the early 1840's. In 1843, William Rowan Hamilton introduced quaternions, which describe mechanics in three-dimensional space. In 1844, Hermann Grassmann published his book *Die lineale Ausdehnungslehre* . Arthur Cayley introduced matrices, one of the most fundamental linear algebraic ideas, in 1857. Despite these early developments, **linear algebra** has been developed primarily in the twentieth century.

Philosophy: Linear Algebra can be seen as a huge collection of techniques and rules that apply to systems of numbers (called vector spaces) rather than to individual numbers. It may not be an accident that human consciousness had to wait for some long for the new concept to emerge and crystallize in the universal mind of humanity. By looking at the big picture linear algebra help us see things which would otherwise be hidden from our awareness. It sometimes takes years of education for new transformed understanding of the world and ourselves to emerge and the process of enlightenment never ends

Applications:

1. On January 1st, 1990 two small trees were planted. The first tree was 4.5 feet tall and grows at a rate of 1.2 feet per month while the second tree was 3 feet tall and grows at a rate of 1.3 feet per month. Approximately, on which date will both trees be the same height?

Answer: The formulas y=4.5 + 1.2*x and y=3+1.5*x represent the heights of the trees as a function of time. Thus, for the heights y to be the same we must solve the equation

4.5 + 1.2*x = 3 + 1.3*x
1.5 = 0.1*x
x=15 (15 month = one year and three month)

Thus the approximate date for the trees to be of the same height would be April 1st, 1991.

Concept: Functions

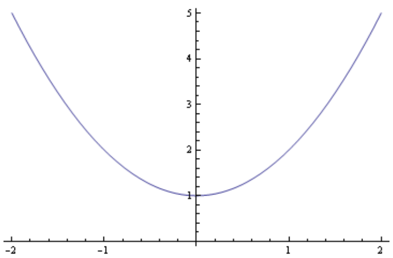
Definition: A **function** expresses *dependence* between two *quantities*: one given (*independent* variable), the other produced (*dependent* variable). Two examples:

1. John's mood is a function of the weather (as well as other factors)

2. The function $y=x^2+1$ can be represented as a table with domain [-1,1] or a graph with domain [-2,2]

x	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	-0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y=x^2+1	2	1.81	1.64	1.49	1.36	1.25	1.16	1.09	1.04	1.01	1	1.01	1.04	1.09	1.16	1.25	1.36	1.49	1.64	1.81	2

Background: Gottfried Leibniz first coined this term in 1694, to describe a curve's slope at a specific point. Later, functions became more and more abstract as mathematics evolved. Modern set theory could define the pairs ((a,x),(b,x)(c,y)) as a function from the set {a,b,c} into the set {x,y,z}.



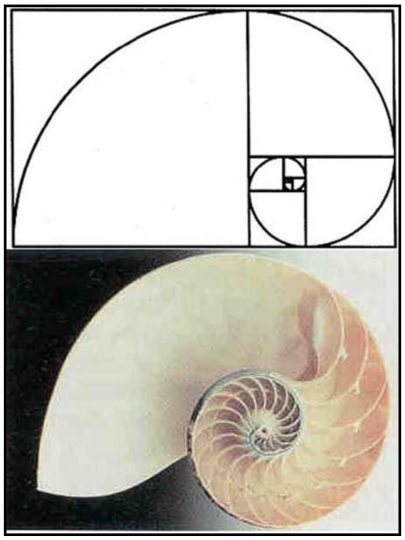
Philosophy: Functions allow us to find patterns not evident to our senses and to discover realities transcending our cultural boundaries. When the Nootkas of Vancouver Island first saw the HMS *Resolution* in 1778, they were convinced Captain Cook's ship was the legendary raven god Yehl. They mistook its prow for a beak and its sails for wings. But after Chief Maquina determined the true function of sails, the tribe realized the bird god was actually a sea vessel like their canoes, only much larger. Even when reality defies our experience and expectations, we can understand the new through analysis and analogy.

Application: If you invest $4000 at an annual rate of 6.0% compounded monthly, what will be its final value after 10 years?

Solution: Taking 6% as the independent variable for the function that computes the final value, we get f(x)=$4000*(1 + x/12)120. Substituting x=6%=0.06 yields the answer $4000*1.005120, which equals $7277.59

Concept: Linear and Exponential Growth

Definition: A **linear function** takes the form f(x)=b∗x+a; an **exponential function** takes the form f(x)=a∗b^x. In both cases, a and b are *constants*.



Remark: Notice the similarity between linear and the exponential functions. While the *difference* is constant in linear cases, the *quotient* is constant in exponential cases. By substituting + → * and * → ^ the linear function becomes exponential.

Background: Among the simplest and most common functions, the linear and the exponential often appear in nature, art and society. For example, the speed of a free-falling object changes linearly, while the shape of the nautilus develops exponentially. The Renaissance masters also based the art of perspective drawing on the exponential function.

Philosophy: Both linear (b*x+a) and exponential (a*b^x) functions depend on two numbers, a and b. If a is the *initial value,* then b is the *rate of change*. Imagine two complementary types of human experience. Let a represent initial conditions at birth and other environmental factors; b can represent individual effort. *Picture a straight line:* Linear growth signifies a life based on constant effort without much creativity. *Now picture a tree*: Exponential growth can represent a rich life in which the present moment inspires creativity to explore branching options.

Applications: Merchant A offers Merchant B this deal. Every day, for the next month, A will give B $10,000; in return, B will give A 1 cent the first day, 2 cents the second, 4 cents the third, and so on—each time doubling the amount. Assuming 30 days in a month, which merchant will profit greater?

Answer: Merchant B's sum will follow linear growth, where f(x)=$10000*x. After 30 days, he will f(30) or $300,000. But Merchant A's sum will grow exponentially. On the first day, he will earn $0.01 on the second day $0.01*2, on the third day $0.01*2^2=$0.04 and so on. By the end of the month, he will have accumulated a fortune of 0.01*2^{29} or $5,368,709.12