Identities and Parseval type relations for the $L_2$-transform

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Abstract

In the present paper, the authors introduce identities for the $L_2$-transform which yield a Parseval type relationship for the $L_2$-transform. The Parseval type relationship proven in this paper give rise to useful corollaries for evaluating indefinite integrals of special functions. Some examples are also given as illustrations of the results presented here.

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1. Introduction

Over a decade ago, Sadek and Yürekli [8] presented a systematic account of so-called the $L_2$-transform:

$$L_2 \{ f(x); y \} = \int_0^\infty x \exp(-x^2y^2)f(x)dx.$$  \hspace{1cm} (1.1)

The $L_2$-transform is related to the classical Laplace transform

$$L \{ f(x); y \} = \int_0^\infty \exp(-xy)f(x)dx$$  \hspace{1cm} (1.2)

via the identities

$$L_2 \{ f(x); y \} = \frac{1}{2} L \{ f(\sqrt{x}); y^2 \},$$  \hspace{1cm} (1.3)

$$L \{ f(x); y \} = 2L_2 \{ f(x^2); \sqrt{y} \}.$$  \hspace{1cm} (1.4)

In subsequent articles, Saygınsoy and Yürekli [4] and Yürekli [6,7], a number of Parseval-Goldstein type of identities involving the $L_2$-transform and various other integral transforms are introduced. In the articles by Wilson and Yürekli [9,10], an alternative method of solving Bessel’s differential equation and Hermite’s differential equation is given using the $L_2$-transform.

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In this paper, after introducing some properties of the $L_2$-transform, we give a Parseval type relationship for the $L_2$-transform. As applications of the identities, some illustrative examples are given.

2. The main theorem

The following properties of $L_2$-transform will be required in our investigation.

**Lemma 1.** The identities

$$L_2\{\exp(-z^2x^2)g(x);y\} = L_2\{g(x);\sqrt{x^2+y^2}\} \quad (2.1)$$

and

$$L_2\{g(\sqrt{x^2-z^2})H(x-z);y\} = \exp(-z^2y^2)\ L_2\{g(x);y\}, \quad (2.2)$$

where $H(x)$ is the Heaviside function, holds true.

**Proof.** Using the definition (1.1) of the $L_2$-transform, we have

$$L_2\{\exp(-z^2x^2)g(x);y\} = \int_0^\infty x \exp(-(z^2+y^2)x^2)g(x)dx. \quad (2.3)$$

Now the assertion (2.1) follows immediately from the definition (1.1) of the $L_2$-transform. To prove (2.2), we use the definitions of the $L_2$-transform and the Heaviside function, and obtain

$$\int_0^\infty x \exp(-x^2y^2)g(\sqrt{x^2-z^2})dx = \int_z^\infty x \exp(-x^2y^2)g(\sqrt{x^2-z^2})dx. \quad (2.4)$$

Changing the variable of integration to $u = \sqrt{x^2-z^2}$, we find

$$L_2\{g(\sqrt{x^2-z^2})H(x-z);y\} = \exp(-z^2y^2)\int_0^\infty u \exp(-u^2y^2)g(u)du. \quad (2.5)$$

Now the assertion (2.2) follows from (1.1). $\Box$

We now state our main result.

**Theorem 2.** The Parseval type relationship

$$\int_0^\infty y^v L_2\{f(x);y\}dy = \frac{1}{2} \Gamma\left(\frac{v+1}{2}\right) \int_0^\infty f(x) x^v dx \quad (2.6)$$

holds true, provided that $\Re(v) > -1$ and the integrals involved converge absolutely.

**Proof.** Using the definition (1.1) of the $L_2$-transform, we have

$$\int_0^\infty y^v L_2\{f(x);y\}dy = \int_0^\infty y^v \left[ \int_0^\infty x \exp(-x^2y^2)f(x)dx \right] dy. \quad (2.7)$$

Changing the order of integration which is permissible under the assumptions of the theorem, we find

$$\int_0^\infty y^v L_2\{f(x);y\}dy = \int_0^\infty xf(x) \left[ \int_0^\infty y^v \exp(-x^2y^2)dy \right] dx. \quad (2.8)$$

Now the assertion (2.6) easily follows when we compute the inner integral on the right-hand side of (2.8). $\Box$

An immediate consequence of Theorem 2 is contained in

**Corollary 3.** If the integrals involved converge absolutely, then we have

$$\int_0^\infty y^v L_2\{g(x);\sqrt{z^2+y^2}\}dy = \frac{1}{2} \Gamma\left(\frac{v+1}{2}\right) L_2\left\{\frac{g(x)}{x^2+1};z\right\} \quad (2.9)$$
and
\[
\int_z^\infty u(u^2 - z^2)^{(v-1)/2} L_2\{g(x); u\}du = \frac{1}{2} \Gamma \left( \frac{v + 1}{2} \right) L_2 \left\{ \frac{g(x)}{x^{v+1}}; z \right\}.
\] (2.10)

**Proof.** We substitute
\[
f(x) = \exp(-z^2x^2)g(x)
\] (2.11)
into (2.6) of Theorem 2 and then utilize the identity (2.1) of Lemma 1. The assertion (2.9) immediately follows upon using the definition (1.1) of the \( L_2 \)-transform. The identity (2.10) follows upon changing the variable of the integration in (2.9) to \( u = z^2 + y^2 \).

**Corollary 4.** Under the assumptions of Theorem 2, we have
\[
L_2\{y^{v-1} L_2\{g(x); y\}; z\} = \frac{1}{2} \Gamma \left( \frac{v + 1}{2} \right) \int_z^\infty \frac{g(\sqrt{x^2 - z^2})}{x^r} dx.
\] (2.12)

**Proof.** We substitute
\[
f(x) = g(\sqrt{x^2 - z^2})
\] (2.13)
into (2.6) of Theorem 2 and then utilize the identity (2.2) of Lemma 1. The assertion (2.12) immediately follows upon using the definition (1.1) of the \( L_2 \)-transform.

**Remark 5.** If we change the variable of the integration in (2.12) to \( u = \sqrt{x^2 - z^2} \), we obtain the identity
\[
L_2\{y^{v-1} L_2\{g(x); y\}; z\} = \frac{1}{2} \Gamma \left( \frac{v + 1}{2} \right) \int_0^\infty \frac{ug(u)}{(u^2 + z^2)^{(v+1)/2}} du.
\] (2.14)
If we set \( v = 1 \) in (2.14), we obtain
\[
L_2\{L_2\{g(x); y\}; z\} = \frac{1}{2} \mathcal{P}\{g(x); y\},
\] (2.15)
where \( \mathcal{P}\{f(x); y\} \) is the Widder potential transform and it is defined as
\[
\mathcal{P}\{f(x); y\} = \int_0^\infty \frac{xf(x)}{x^2 + y^2} dx.
\] (2.16)
The identity (2.15) is obtained earlier in Sadek and Yürekli [8, Eq. (2.1), p. 518]. Therefore, the relationship (2.14) is a generalization of the earlier identity (2.15).

3. Illustrative examples

An illustration of the Parseval type relation (2.6) of Theorem 2 is given in the following example.

**Example 6.** We have
\[
\int_0^\infty \frac{\text{Erf}(ax)}{x^r} dx = \frac{a^{r-1}}{\sqrt{\pi(v - 1)}} \Gamma \left( 1 - \frac{v}{2} \right),
\] (3.1)
where \(-1 < \Re(v) < 0\).

**Proof.** We set
\[
f(x) = \text{Erf}(ax)
\] (3.2)
in the identity (2.6) of Theorem 2. Using the relationship (1.3) and [1, Entry (4), p. 176], we obtain
we obtain
\[ \mathcal{L}_2\{\text{Erf}(ax); y\} = \frac{1}{2} \mathcal{L}\{\text{Erf}(ax^{1/2}); y^2\}, \quad (3.3) \]
\[ \frac{a}{2y^2(y^2 + a^2)^{1/2}}. \quad (3.4) \]
Substituting Eqs. (3.2) and (3.4) into the identity (2.6), we obtain
\[ \int_0^\infty \frac{\text{Erf}(ax)}{x^v} \, dx = a \left[ \Gamma\left(\frac{v + 1}{2}\right)\right]^{-1} \int_0^\infty \frac{y^{v-2}}{(y^2 + a^2)^{1/2}} \, dy. \quad (3.5) \]
We utilize the integral representation for the Gamma function (cf. [2, p. 7])
\[ \int_0^\infty t^v \, dt = \frac{\Gamma(x + 1)\Gamma(y - x)}{\Gamma(1 + y)}, \quad (3.6) \]
where \( \Re(y) > \Re(x) > -1 \), to evaluate the integral on the right-hand side of (3.5) and obtain the assertion (3.1).

**Example 7.** We have
\[ \int_0^\infty x^{v-1} \exp(-a^2 x^2) \, dx = \frac{1}{2a^v} \Gamma\left(\frac{v}{2}\right), \quad (3.7) \]
where \( \Re(v) > 0 \).

**Proof.** We set
\[ f(x) = x^{-1} \exp\left(-\frac{a^2}{4x^2}\right) \quad (3.8) \]
in the identity (2.6) of Theorem 2. Using the relationship (1.3) and [1, Entry (27), p. 146], we obtain
\[ \mathcal{L}_2\left\{x^{-1} \exp\left(-\frac{a^2}{4x^2}\right); y\right\} = \frac{1}{2} \mathcal{L}\left\{x^{-1/2} \exp\left(-\frac{a^2}{4x}\right); y^2\right\} = \frac{\sqrt{\pi}}{2y} \exp(-ay). \quad (3.9) \]
Substituting Eqs. (3.8) and (3.9) into the identity (2.6), we obtain
\[ \int_0^\infty x^{v-1} \exp\left(-\frac{a^2}{4x^2}\right) \, dx = \sqrt{\pi} \left[ \Gamma\left(\frac{v + 1}{2}\right)\right]^{-1} \int_0^\infty y^{v-1} \exp(-ay) \, dy. \quad (3.10) \]
The integral on the right-hand side is the Laplace transform of \( y^{v-1} \) which has the value
\[ \mathcal{L}\{y^{v-1}; a\} = \frac{\Gamma(v)}{a^v}, \quad (3.11) \]
where \( \Re(v) > 0 \). Setting \( z = v/2 \) in the formula for the \( \Gamma \)-function
\[ \Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma\left(\frac{1}{2} + z\right), \quad (3.12) \]
we obtain
\[ \Gamma(v) = \frac{2^{v-1}}{\sqrt{\pi}} \Gamma\left(\frac{v}{2}\right)\Gamma\left(\frac{1}{2} + \frac{v}{2}\right). \quad (3.13) \]
Substituting the formulas (3.11) and (3.13) into Eq. (3.10), we find
\[ \int_0^\infty x^{v-1} \exp\left(-\frac{a^2}{4x^2}\right) \, dx = \frac{1}{2a^v} \Gamma\left(\frac{v}{2}\right). \quad (3.14) \]
Now the assertion follows when we change the variable of the integration on the left-hand side of (3.14) to \( u = 1/2x \).
Remark 8. If we put \( v = 1 \) in (3.7) of Example 7, we obtain the well known result
\[
\int_0^\infty \exp(-a^2x^2)dx = \frac{\sqrt{\pi}}{2a}.
\]
(3.15)

Remark 9. If we put \( v - 1 = \mu \) in (3.7) of Example 7 and use the definition of (1.1) of the \( \mathcal{L}_2 \)-transform, we deduce
\[
\mathcal{L}_2\{\mu+1/2\} = \frac{1}{2} I\left(\frac{\mu+1}{2}\right) \frac{1}{2} y^{\mu+1},
\]
(3.16)
provided that \( \Re(\mu) > -1 \).

Example 10. We have
\[
\int_0^\infty y^{v}(y^2 + z^2)^{-1/2} \exp[-a(y^2 + z^2)^{1/2}]dy = \pi^{-1/2} 2^{v/2} I\left(\frac{v+1}{2}\right) \left(\frac{2z}{a}\right)^{v/2} K_{v/2}(az)
\]
(3.17)
and
\[
\int_z^\infty (u^2 - z^2)^{(v-1)/2} \exp(-au)du = \pi^{-1/2} 2^{v/2} I\left(\frac{v+1}{2}\right) \left(\frac{2z}{a}\right)^{v/2} K_{v/2}(az),
\]
(3.18)
where \( \Re(v) > -1 \) and \( K_v(x) \) is the modified Bessel function of the second kind.

Proof. We set
\[
g(x) = \frac{1}{x} \exp\left(-\frac{a^2}{4x^2}\right)
\]
(3.19)
in the identity (2.9) of Corollary 3. Using (3.9), we have
\[
\mathcal{L}_2\{g(x); (y^2 + z^2)^{1/2}\} = \frac{\pi^{1/2}}{2} (y^2 + z^2)^{-1/2} \exp[-a(y^2 + z^2)^{1/2}].
\]
(3.20)
Using the identity (1.3) and then the formula [1, Entry (29), p. 146], we deduce
\[
\mathcal{L}_2\{x^{-v/2} \exp(-a^2/4x^2); z\} = \frac{1}{2} \mathcal{L}_2\{x^{-(v/2)-1} \exp(-a^2/4x^2); z^2\} = \left(\frac{2z}{a}\right)^{v/2} K_{v/2}(az).
\]
(3.21)
Now the assertion (3.17) follows when we substitute the results (3.19)–(3.21) into (2.9). Similarly, the assertion (3.18) follows when we substitute the results (3.19), (3.9) and (3.21) into (2.10). \( \square \)

Remark 11. If we put \( v/2 = \mu \) in (3.17) of Example 10 and solve the Eq. (3.17) for \( K_\mu(az) \), we deduce
\[
K_\mu(az) = \frac{\sqrt{\pi}}{I(\mu+1/2)} \left(\frac{a}{2z}\right)^{\mu} \int_0^\infty \frac{y^{2\mu}}{(y^2 + z^2)^{1/2}} \exp\left[-az\left(\frac{y^2}{z^2} + 1\right)^{1/2}\right]dy.
\]
(3.22)
Changing the variable of integration on the right-hand side of (3.22) to \( t = y/z \) and setting \( az = x \), we obtain the following integral representation for the modified Bessel function of the second kind \( K_\mu(x) \):
\[
K_\mu(x) = \frac{\sqrt{\pi}}{I(\mu+1/2)} \left(\frac{x}{2}\right)^{\mu} \int_0^\infty \frac{t^{2\mu}}{(t^2 + 1)^{1/2}} \exp[-x(t^2 + 1)^{1/2}]dt,
\]
(3.23)
where \( \Re(\mu) > -1/2 \). Similarly, we deduce another well known integral representation for the function \( K_v(x) \) from the result (3.18) of Example 10:
\[
K_\mu(x) = \frac{\sqrt{\pi}}{I(\mu+1/2)} \left(\frac{x}{2}\right)^{\mu} \int_1^\infty \exp\left(-xt\right) \frac{1}{(t^2 - 1)^{1/2-\mu}}dt,
\]
(3.24)
where \( \Re(\mu) > -1/2 \) (cf. [5, Entry (51:3:3), p. 500]).
Example 12. We have
\[ \int_0^\infty y^v \exp(a^2 y^2) \text{Erfc}(ay) dy = \frac{1}{a^{v+1}} \Gamma\left(\frac{v+1}{2}\right) \csc\left(\frac{v\pi}{2}\right) \]
where \(-2 < \Re(v) < 0\).

Proof. We set
\[ f(x) = \frac{1}{x(x^2 + a^2)} \]
in the identity (2.6) of Theorem 2. Using the relationship (1.3) and [3, Entry (3), p. 16], we obtain
\[ \mathcal{P}\left\{ \frac{1}{x(x^2 + a^2)} ; y \right\} = \frac{1}{2} \mathcal{P}\left\{ x^{-1/2} \frac{x^2 + a^2}{x + a^2} ; y \right\} = \frac{\pi}{2a} \exp(-a^2 y^2) \text{Erfc}(ay). \]
Substituting Eqs. (3.26) and (3.27) into the identity (2.6), we obtain
\[ \int_0^\infty y^v \exp(a^2 y^2) \text{Erfc}(ay) dy = \frac{a}{\pi} \Gamma\left(\frac{v+1}{2}\right) \int_0^\infty x^{-v-1} \frac{x^2 + a^2}{x^2 + a^2} dx. \]
Using the definition (2.16) of the Widder potential transform on the right-hand side of (3.28), we obtain
\[ \int_0^\infty y^v \exp(a^2 y^2) \text{Erfc}(ay) dy = \frac{a}{\pi} \Gamma\left(\frac{v+1}{2}\right) \mathcal{P}\{x^{-v-2} ; y\}. \]
Using the formula [8, (A1), p. 248] for the Widder potential transform on the right-hand side of (3.29) and using some results on trigonometric functions, we obtain the assertion (3.25).

Example 13. We have
\[ \int_0^\infty \frac{\cos(ax^2)}{x^v} dx = \frac{1}{2} a^{(v-1)/2} \cos \left[ \frac{\pi(1-v)}{4} \right] \Gamma\left(\frac{1-v}{2}\right) \]
and
\[ \int_0^\infty \frac{\sin(ax^2)}{x^v} dx = \frac{1}{2} a^{(v-1)/2} \cos \left[ \frac{\pi(1+v)}{4} \right] \Gamma\left(\frac{1-v}{2}\right), \]
where \(-1 < \Re(v) < 1\).

Proof. We set
\[ f(x) = \cos(ax^2) \]
in identity (2.6) of Theorem 2. Using the relationship (1.3) and [1, Entry (43), p. 154], we obtain
\[ \mathcal{P}\{\cos(ax^2) ; y\} = \frac{1}{2} \mathcal{P}\{\cos(ax) ; y^2\} = \frac{1}{2} \frac{y^2}{a^2 + y^2}. \]
Substituting Eqs. (3.32) and (3.33) into the identity (2.6), we obtain
\[ \int_0^\infty \frac{\cos(ax^2)}{x^v} dx = \left[ \Gamma\left(\frac{v+1}{2}\right) \right]^{-1} \int_0^\infty \frac{y^{v+2}}{a^2 + y^2} dy. \]
Using the integral representation (3.6) and the well-known identity
\[ \Gamma(z)\Gamma(1-z) = \pi \csc(\pi z) \]
on the right-hand side of (3.34), we obtain (3.30).
We obtain (3.31) in a similar fashion by substituting \(f(x) = \sin(ax^2)\) into identity (2.6).
Example 14. We have
\[
\int_0^\infty \frac{y^\nu}{(y^2+z^2)^2+a^2} \, dy = \frac{\pi}{2a} \csc \left( \frac{\pi \nu}{2} \right) (z^4 + a^2)^{\nu-1/4} \sin \left[ \frac{1-v}{2} \arctan \left( \frac{a}{z^2} \right) \right],
\]
where \( \Re(v) < 3 \).

\textbf{Proof.} Setting 
\[ g(x) = \sin(ax^2) \]
in identity (2.9) of Corollary 3. Using the identity (1.3), we have
\[
\mathcal{L}_2 \{ \sin(ax^2); (y^2+z^2)^{1/2} \} = \frac{1}{2} \mathcal{L} \{ \sin(ax); y^2+z^2 \} = \frac{a}{2} \frac{1}{(z^2+y^2)^2+a^2}. \]

Using the identity (1.3) and then the formula [1, Entry (15), p. 152], we deduce
\[
\mathcal{L}_2 \{ x^{-\nu-1} \sin(ax^2); z \} = \frac{1}{2} \mathcal{L} \{ x^{-(\nu-1)/2} \sin(ax); z^2 \} = \frac{1}{2} \Gamma \left( \frac{1-v}{2} \right) (z^4 + a^2)^{(\nu-1)/4} \sin \left[ \frac{1-v}{2} \arctan \left( \frac{a}{z^2} \right) \right].
\]

Substituting (3.36)–(3.38) into (2.9) yields
\[
\int_0^\infty \frac{y^\nu}{(y^2+z^2)^2+a^2} \, dy = \frac{1}{2a} \Gamma \left( \frac{\nu+1}{2} \right) \Gamma \left( \frac{1-v}{2} \right) \cdot (z^4 + a^2)^{(\nu-1)/4} \sin \left[ \frac{1-v}{2} \arctan \left( \frac{a}{z^2} \right) \right].
\]

Using the identity
\[
\Gamma \left( \frac{\nu+1}{2} \right) \Gamma \left( \frac{1-v}{2} \right) = \pi \csc \left( \frac{\pi \nu}{2} \right)
\]
on the right-hand side of (3.39) gives the desired result.

We conclude that many other infinite integrals can be evaluated in this manner by applying the lemma, the theorem and its corollaries considered here.

\textbf{References}