Geometric properties of three-dimensional fractal trees

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1. Introduction

A fractal tree defined by an iterated function system can be loosely defined as a trunk and a number of branches that each look like the tree itself. Often, these appear strikingly similar to real trees, and hence are used frequently as tree models and models for lung growth. Fractal tree models have also been used in other areas, such as antenna construction [2] and mantle melting [3]. Many other applications of the structure of fractal trees are found in [6–10].

Mathematically, these trees have been studied primarily by Mandelbrot and Frame [4,5], who proved and conjectured properties of symmetric binary fractal trees in the plane. Mandelbrot and Frame looked into plane-filling trees, the shape of the canopy (or tip set), and conditions for self-contacting trees. Less work has been done concerning fractal trees in three dimensions, though some recent work is found in [1]. This work describes self-contact in symmetric three-dimensional fractal trees. The authors also relate this self-contact condition to connectedness of the attractors of the tree iterated function system.

We also mention that the search for scaling that results in tip-to-tip contact involves rather interesting constants, depending, of course, on the branching angles. In two-dimensional trees with two branchings at angle of $\pi/3$, we see that the required scaling ratio turns out to be $1/\phi$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio [4]. We see this same scaling ratio required in three-dimensional trees when the branching angle is $\arccos(-1/\sqrt{3})$, see [1]. The appearance of the golden mean in the study of fractals and Hausdorff dimension can be found in [11–13].

In this paper, we relate the study of geometric properties of the trees (length, area, and volume) to the well-known measure of fractal dimension coming from Moran’s equation. Recall that Moran’s equation computes the fractal dimension, $D$, of a fractal with scaling ratios $r_1, r_2, \ldots, r_N$ via the solution to

$$r_1^0 + r_2^0 + \cdots + r_N^0 = 1.$$  \hspace{1cm} (1)

Here, we make a few remarks. In this paper, we consider trees that are symmetric in their scaling; that is, all scaling ratios are equal. Hence, a tree with $b$ branches and scaling ratio $r$ reduces (1) to

$$b r^D = 1.$$ 

Also, the dimension $D$ refers to the dimension of the resulting attractor of the fractal tree (also known as the canopy) and not of the branches themselves, which of course always have dimension 1.

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2. Fractal trees

First, we define fractal trees more rigorously, as the result of an iterated function system applied to the trunk of a tree (in our case, the line segment connecting the origin with \((0,0,1)\)).

**Definition 1** (fractal tree). We denote a three-dimensional fractal tree by \( F = F(r, \varphi, b) \), where \( r \) is the scaling ratio at each level, \( \varphi \) is the angle of rotation in the \( y-z \) plane, and \( b \) is the number of branches.

With these parameters, we can construct the tree using affine transformations.

**Definition 2** (tree transformations). For a fractal tree \( F = F(r, \varphi, b) \), the corresponding affine transformations are

\[
T_j(x) = \tilde{r} + r \begin{bmatrix}
\cos(\theta_j) & -\sin(\theta_j) & 0 \\
\sin(\theta_j) & \cos(\theta_j) & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\
0 & \cos(\varphi) & -\sin(\varphi) \\
0 & \sin(\varphi) & \cos(\varphi)
\end{bmatrix} \begin{bmatrix} x \\end{bmatrix},
\]

where \( \theta_j = 2\pi(j - 1)/b \) and \( 1 \leq j \leq b \). The trunk \( \tilde{r} \) of the tree is assumed throughout to be the standard vector \( \tilde{e}_3 = \begin{bmatrix} 0 \\
0 \\
1 \end{bmatrix} \).

**Definition 3** (tip set). Let \( F = F(r, \varphi, b) \) be a fractal tree. The tip set or canopy of \( F \), denoted \( \text{tip}(F) \), is the attractor of the iterated function system \( \{T_1, \ldots, T_b\} \). Thus, \( \text{tip}(F) \) is the fixed point of the function \( F : \mathcal{H} \to \mathcal{H} \), where \( F(S) = T_1(S) \cup \cdots \cup T_b(S) \):

\[
F(\text{tip}(F)) = \bigcup_{i=1}^b T_i(\text{tip}(F)) = \text{tip}(F).
\]

Fig. 2 shows the canopy of the fractal tree \( F = F(\frac{1}{2}, \frac{\pi}{3}, 3) \). Notice that the branches of the tree grow to form the canopy.

3. Length

Lengths in fractal trees have been studied previously \[14\] and bear the most direct link to fractal dimension. Consider a three-branch tree, as in Fig. 1. The total length, \( \mathcal{L} \), encompassed by the tree is the sum of all the lengths at each level:

\[
\mathcal{L} = 1 + br + b^2r^2 + b^3r^3 + \cdots = \sum_{k=0}^{\infty} (br)^k = \frac{1}{1-br},
\]

if \( r < 1/b \). So, if the scaling ratio is less than the reciprocal of the number of branchings then the tree has a total finite length.

Recalling the definition of fractal dimension, the critical value of \( r = 1/b \) corresponds to the solution of Moran’s equation for dimension \( 1 : br = 1 \). This also gives the minimal possible condition for connectedness of the canopy of the tree (though it does not guarantee connectedness). Hence, all trees with finite total length must be self-avoiding; that is, they do not possess connected canopies, and in fact, the canopies are totally disconnected.

4. Area of a petal

Consider a three branch three-dimensional fractal tree, and imagine, at each level of the tree, that each we consider the quadrilateral generated by sequential pairs of branches, see Fig. 3. The area of each petal can be found by computing the

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**Fig. 1.** The construction of the \( F = F(\frac{1}{2}, \frac{\pi}{3}, 3) \) fractal tree.
norm of the cross product of the vectors generating the quadrilateral. By the fractal self-similarity, the total area encompassed by these petals can be determined once we determine the area of the petals at the first level of construction.

Referring to Fig. 4, we see that the area of any level one petal can be computed using the vectors

\[ \vec{V}_1 = T_1(0, 0, 1) - (0, 0, 1) \quad \text{and} \quad \vec{V}_2 = T_2(0, 0, 1) - (0, 0, 1). \]

In this particular example, our tree is a three-branch tree, and so, \( \theta_1 = 0 \) and \( \theta_2 = 2\pi/3 \). Thus

\[ \vec{V}_1 = \begin{pmatrix} 0 \\ -r \sin \phi \\ r \cos \phi \end{pmatrix} \quad \text{and} \quad \vec{V}_2 = \begin{pmatrix} r \sin \phi \sin(2\pi/3) \\ -r \sin \phi \cos(2\pi/3) \\ r \cos \phi \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} r \sin \phi \\ \frac{1}{2} r \sin \phi \\ r \cos \phi \end{pmatrix} \]
Then, the area of each of the level one petals is

\[ a_1 = \|\mathbf{V}_1 \times \mathbf{V}_2\| = r^2 \sin \phi \sqrt{3 \cos^2 \phi + (3/4) \sin^2 \phi} = \frac{3}{2} r^2 \sqrt{\sin^4 \phi + \sin^2(2\phi)} \]

and the total area at level one is \( A_1 = 3a_1 \). At the next iteration, we find 9 petals, which are self-similar to the level one petals, except that the side lengths are now \( r^2 \), and hence the total area at level two is

\[ A_2 = 3^2 \frac{\sqrt{3}}{2} (r^2)^2 \sqrt{\sin^4 \phi + \sin^2(2\phi)}. \]

Continuing, we find that the total petal area at the \( n \)th iteration is

\[ A_n = 3^n \frac{\sqrt{3}}{2} (r^n)^2 \sqrt{\sin^4 \phi + \sin^2(2\phi)} = \left( \frac{\sqrt{3}}{2} \sqrt{\sin^4 \phi + \sin^2(2\phi)} \right) 3^n r^{2n}. \]

Now, we wish to find the total area, \( \mathcal{A} \), encompassed by the petals on the tree. By summing all the iterations we find a geometric series, which is easily summable.

\[
\mathcal{A} = \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \left( \frac{\sqrt{3}}{2} \sqrt{\sin^4 \phi + \sin^2(2\phi)} \right) 3^n r^{2n} = \left( \frac{\sqrt{3}}{2} \sqrt{\sin^4 \phi + \sin^2(2\phi)} \right) \frac{1}{1 - 3r^2},
\]

which converges if \( 3r^2 < 1 \) or \( r < 1/\sqrt{3} \).

The general case follows in a similar fashion, yielding the following.

**Theorem 4.** Given a symmetric fractal tree, \( \mathcal{F}(r, \phi, b) \), with \( b \) branches and scaling ratio \( r \), the total area encompassed by the petals formed from consecutive branches at each level is finite if and only if \( r < 1/\sqrt{b} \), and this area is given by

\[
\mathcal{A} = \frac{(\sin \phi) \sqrt{2 \cos^2 \phi + \sin^2 \phi \sin^2(2\pi/b) - 2 \cos^2 \phi \cos(2\pi/b)}}{1 - br^2}.
\]

**Proof 1.** The area of a single level one petals is generated by the branch vectors

\[
\mathbf{V}_1 = \begin{pmatrix} 0 \\ -r \sin \phi \\ r \cos \phi \end{pmatrix} \quad \text{and} \quad \mathbf{V}_2 = \begin{pmatrix} r \sin \phi \sin(2\pi/b) \\ -r \sin \phi \cos(2\pi/b) \\ r \cos \phi \end{pmatrix},
\]

and this area is

\[ a_1 = \|\mathbf{V}_1 \times \mathbf{V}_2\| = r^2 \sin \phi \sqrt{2 \cos^2 \phi + \sin^2 \phi \sin^2(2\pi/b) - 2 \cos^2 \phi \cos(2\pi/b)}. \]

Then, the total area at level one is

\[ A_1 = br^2 \sin \phi \sqrt{2 \cos^2 \phi + \sin^2 \phi \sin^2(2\pi/b) - 2 \cos^2 \phi \cos(2\pi/b)} \]

and the total area at level \( n \) is

\[ A_n = b^n (r^n)^2 \sin \phi \sqrt{2 \cos^2 \phi + \sin^2 \phi \sin^2(2\pi/b) - 2 \cos^2 \phi \cos(2\pi/b)}. \]

So, the total area is

\[
\mathcal{A} = \sum_{n=0}^{\infty} A_n = \left( \sin \phi \sqrt{2 \cos^2 \phi + \sin^2 \phi \sin^2(2\pi/b) - 2 \cos^2 \phi \cos(2\pi/b)} \right) \sum_{n=0}^{\infty} b^n r^{2n} = \frac{\sin \phi \sqrt{2 \cos^2 \phi + \sin^2 \phi \sin^2(2\pi/b) - 2 \cos^2 \phi \cos(2\pi/b)}}{1 - br^2}.
\]

Given this theorem, the link to the canopy dimension is now obvious.

**Corollary 5.** The scaling ratio, \( r \), that determines the limit of finite area encompassed by the branches of a fractal tree \( \mathcal{F}(r, \phi, b) \) is the value of \( r \) that determines that the canopy dimension of the tree is 2, namely \( r = 1/\sqrt{b} \).
5. Volume of flowers

Now, consider putting the petals together to form “flowers” in the tree, see Fig. 5. See Fig. 6 for the flower at the first iteration. Here, the petals are triangular and each flower petal has half the area of the quadrilateral petals in Section 4. (Note that using half quadrilaterals does not change the result on the critical scaling ratio for finite area in the previous section.)

The volume of any such flower (with $b$ petals) is easily computed as the volume of a pyramid with polygonal base (with $b$ sides). In particular, first compute the volume of the level one flower, which is generated by the vectors $\tilde{V}_j = T_j(0, 0, 1) - (0, 0, 1), 1 \leq j \leq b$. The resulting flower in a tree $\mathcal{F}(r, 0, b)$ is a pyramid with base area $A_{\text{base}} = b(1/2)r^2 \sin^2 \phi \sin(2\pi/b)$, see Fig. 7.

The base radius $R$ is given by the projection of $\tilde{V}_2$ onto the $xy$-plane. So, $R = r \sin \phi$, and the base area follows. The height of the pyramid is given by the $z$-coordinate of $\tilde{V}_2$, which is $h = r \cos \phi$. Thus, the level one volume is

$$V_1 = \frac{1}{3} A_{\text{base}} h = \frac{1}{6} br^3 \sin^2 \phi \cos \phi \sin(2\pi/b).$$

Now, to find the total flower volume at level $n$, we note that the self-similarity of the fractal tree allows us to compute the volume of a single level $n$ flower by replacing the side length $r$ with $r^n$. Since there are $b^{n-1}$ flowers at level $n$, we find that the total volume is
\[ V_n = b^n \left( \frac{1}{6} b (r^n)^3 \sin^2 \phi \cos \phi \sin(2\pi/b) \right) = \frac{1}{6} b^n r^{3n} \sin^2 \phi \cos \phi \sin(2\pi/b). \]

Now, the total tree flower volume is

\[ V = \sum_{n=0}^{\infty} V_n = \sum_{n=0}^{\infty} \frac{1}{6} b^n r^{3n} \sin^2 \phi \cos \phi \sin(2\pi/b) = \frac{1}{6} \sin^2 \phi \cos \phi \sin(2\pi/b) \sum_{n=0}^{\infty} (br^3)^n \]

\[ = \frac{1}{6} \sin^2 \phi \cos \phi \sin(2\pi/b) \frac{1}{1 - br^3}, \]

which is finite if and only if \( r < \frac{1}{\sqrt[3]{b}} \). Thus, we have proved.

**Theorem 6.** Given a symmetric fractal tree, \( T(r, \phi, b) \), with \( b \) branches and scaling ratio \( r \), the total volume encompassed by the flowers formed from consecutive branches at each level is finite if and only if \( r < \frac{1}{\sqrt[3]{b}} \), and this volume is given by

\[ V = \frac{1}{6} \sin^2 \phi \cos \phi \sin(2\pi/b) \frac{1}{1 - br^3}. \]

Further, \( r = \frac{1}{\sqrt[3]{b}} \) is the critical scaling ratio for which the dimension of the tree canopy is 3.

6. Conclusion

In this paper, we have shown that there is a natural connection between the dimension of the attractor of a fractal tree iterated function system and geometric constructions within the tree branches (which are not part of the attractor).

References